Center for Theoretical Physics (CTN)
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WRITTEN EXAMINATION RELATIVISTIC QUANTUM MECHANICS

Friday 05-04-2013, 09.00-12.00

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 16 parts. The 16 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

Consider the following Lagrangian density for a quantum field theory involving two scalar fields ϕ_1 and ϕ_2:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + a\partial_\mu\phi_1\partial^\mu\phi_2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - 2m^2\phi_1\phi_2\,, \tag{1}$$

where a is a constant and $m \neq 0$ a mass parameter.

1.1 Determine the equations of motion for ϕ_1 and ϕ_2.

1.2 What are the canonical momenta π_1 and π_2 associated to ϕ_1 and ϕ_2?

1.3 Express the Hamiltonian, for the special case $a = 0$, in terms of the canonical momenta and spatial derivatives of the fields.

1.4 Show that for $a = 1$ the equations of motion imply that $\phi_1 = \phi_2$. What is in this case the mass of the field $\phi \equiv \phi_1 + \phi_2$?

PROBLEM 2

Consider a massive vector field $A_\mu(x)$, with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2 A_\mu A^\mu\,, \tag{2}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and m a mass parameter.

2.1 Determine the equations of motion for $A_\mu(x)$.

2.2 Show that the equations of motion, derived in (2.1), imply that

$$\left(\partial_\mu \partial^\mu + m^2\right) A^\nu = 0\,, \qquad \partial_\mu A^\mu = 0\,. \tag{3}$$

2.3 Consider the limit $m \longrightarrow 0$ in the Lagrangian density (2). Is it allowed to take this limit in the equations of motion (3)? Explain!

2.4 Explain how the constraint $\partial_\mu A^\mu = 0$ can arise from the $m \longrightarrow 0$ limit of the Lagrangian (2).

PROBLEM 3

Consider the theory of a scalar field $\phi(x)$, with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) - \tfrac{1}{2}m^2 \phi(x)\phi(x).$$

3.1 What is the canonical momentum $\pi(x)$ corresponding to the coordinate $\phi(x)$?

3.2 Given that classically $\{\phi(t,\vec{x}), \phi(t,\vec{y})\}_{\mathrm{PB}} = 0$, what is the result of the equal-time commutation relation

$$[\phi(x), \phi(y)]_{x^0=y^0} \tag{4}$$

for the quantum operator $\phi(x)$?

3.3 The operator $\phi(x)$ can be written in the form

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left(a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{+ikx}\right)_{k^0=\omega_k}. \tag{5}$$

Show that (for $x^0 \neq y^0$!)

$$[\phi(x), \phi(y)]_{x^0 \neq y^0} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik(x-y)} - e^{+ik(x-y)}\right)_{k^0=\omega_k} . \tag{6}$$

delta functions?

3.4 Using (3.3) show that in the limit $x^0 \to y^0$ the result of (3.2) is obtained.

PROBLEM 4

In a scattering process the particles in the initial and final state are considered to be free relativistic particles.

Consider the elastic scattering process between two electrons:

$$e_1 + e_2 \longrightarrow e_3 + e_4 , \tag{7}$$

with four-momenta k_1^μ, k_2^μ, k_3^μ, k_4^μ. The spatial momenta satisfy (center of mass frame)

$$\vec{k}_1 + \vec{k}_2 = 0 . \tag{8}$$

The electrons have mass m.

4.1 What is the value of $(k_i)^\mu (k_i)_\mu$ for $i = 1, \ldots, 4$ (no summation over i)?

4.2 What is the value of $\vec{k}_3 + \vec{k}_4$?

4.3 Show that the energies of the four electrons are equal.

4.4 Choose a coordinate system such that the spatial momentum of e_1 is

$$\vec{k}_1 = (k, 0, 0) . \tag{9}$$

We now perform a Lorentz boost in the x^1 direction. On the momenta this acts as on the coordinates:

$$k_i^{0\prime} = \gamma(k_i^0 - v k_i^1), \quad k_i^{1\prime} = \gamma(-v k_i^0 + k_i^1), \quad k_i^{2\prime} = k_i^2, \quad k_i^{3\prime} = k_i^3, \tag{10}$$

where the index $i = 1, \ldots, 4$ indicates the four electrons and $\gamma = 1/\sqrt{1 - v^2}$. Calculate $\vec{k}_3{}' + \vec{k}_4{}'$ in this new coordinate system.